2010

Thermodynamics of Modified Theories of Gravity

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by

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A thesis presented to the Department of Physics
in partial fulfillment of the requirements for the degree of

Masters of Science

in the subject of Physics

Virginia Commonwealth University

Richmond, Virginia

May 2010

Abstract

Einstein's equations are derived by following Jacobson's thermodynamic method. It is seen that a family of possible field equations exist which satisfy the thermodynamic argument. Modified theories of gravity are addressed as possible candidates for replacing dark matter as an explanation for anomalous cosmological phenomena. Many of the proposed modified theories are not powerful enough to explain the currently observed phenomena and are rejected as viable theories of gravity. A surviving candidate, TeVeS, is further analyzed under the aforementioned thermodynamic argument to check for its consistency with thermodynamics.

Acknowledgements

I would foremost like to thank Dr. Robert Gowdy for taking the time to teach me general relativity and for sharing his thoughts on theoretical physics. I thank Joseph Gordon for all of his invaluable insight and for partaking on this journey with me. Special thanks are given to Aaron Fried for inspiring in me a sense of passion for the mathematical sciences and Craig Rogers for teaching me so much.

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1. Introduction

It is now believed by many contemporary physicists that there is a fundamental connection between general relativity and thermodynamics, but this has not always been the case. Research in classical relativity theory prior to the 1970's had revealed a series of laws which seemed to govern the mechanics of black holes [1, pg-92], which are as follows:

Zeroth Law -The event horizon of a stationary black hole (at equilibrium) has constant surface gravity.

First Law

$$dM = \frac{k}{8\pi} dA + \Omega dJ + \Phi dQ \,,$$

where M is the mass, A is the horizon surface area, Ω is the angular velocity, J is the angular momentum, Φ is the electrostatic potential, Q is the electric charge, and k is the surface gravity of the horizon.

Second Law - The horizon area never decreases.

Third Law - A black hole cannot have vanishing surface gravity.

These four laws have a peculiar similarity to the four laws of thermodynamics. The zeroth law parallels its thermodynamic counterpart in that, temperature is constant in a body at thermal equilibrium. Like wise, the first law in both the black hole and thermodynamic paradigms is simply an expression of conservation of energy, where mass is linked to energy via

$$E = Mc^2 . \tag{1.1}$$

The second law of black hole mechanics is analogous to that of thermodynamics, with horizon surface area replacing entropy. Finally, the third law stipulates that absolute zero of surface gravity cannot be reached. At this point the laws of black hole mechanics are derived from strictly general relativistic considerations and happen to oddly parallel those of thermodynamics.

In 1972, Jacob Bekenstein [2] became very concerned with a certain inconsistency in the thermodynamics around black holes. Consider an observer outside the event horizon of a black hole. The observer throws a container of hot gas past the event horizon. For all intensive purposes the entropy of the container and its contents has vanished from all possible observation. Since entropy effectively decreases, it is a direct violation of the second law of thermodynamics. Bekenstein concluded that thermodynamics must be preserved and black holes themselves must have entropy. He went as far as determining that a black hole's entropy must be proportional to its horizon area, however, he wasn't able to determine the exact relationship. Shortly after, Stephen Hawking was not only able to determine the exact relationship between the entropy and the horizon area, but he also concluded that black holes have an associated temperature [3]. This was enough to conclude that black holes are indeed thermodynamic bodies and are subject to the laws of thermodynamics. However, black holes are only one facet of general relativity. There was not yet a complete connection between general relativity and thermodynamics.

Following this, it was realized that any accelerated observer experiences an effective event horizon which is subject to the same entropy argument made for black holes. Furthermore, any such accelerated observer also experiences a temperature associated with that acceleration. Finally, in 1995 Ted Jacobson [4] was able to derive the Einstein field equations from the thermodynamic relationship

$$dQ = TdS \, .$$

At that point the connection between thermodynamics and general relativity had been solidified. This connection can be pushed farther by imposing the condition that any metric theory of gravity must be consistent with thermodynamics. This results in not just the Einstein equations, but a family of possible field equations. From this set, we can probe for solutions that account for phenomena which Einstein's equations handle awkwardly or not at all. For instance, a new field equation may be able to account for the peculiar stellar velocity distribution of spiral galaxies without having to appeal to dark matter.

2. Quantum Field Theory

The goal of quantum field theory is to create a quantum mechanical interpretation of the fields of classical physics. This is a daunting task which is often simplified by considering a classical model. Consider a rod of masses each connected by a series of springs which are only displaced in the horizontal direction [5, pg-558].

Let the equilibrium position of each mass be q_{0i}. If the masses are shifted from the equilibrium position by η_i then the position q_i of each mass is

$$q_i = q_{0i} + \eta_i.$$

If the mass of each particle is the same, the kinetic energy of the system is then

$$T = \frac{1}{2}\sum_i m\left(\dot{q}_i\right)^2 = \frac{1}{2}\sum_i m\left(\dot{\eta}_i\right)^2.$$

The potential energy is given by the sum of the potential energies stored in each spring. The potential energy of each spring is determined by its extension from its equilibrium position, which is dependent only on the position of neighboring masses. If η_i represents a particular mass's deviation from equilibrium and η_{i+1} represents the deviation of its nearest neighbor from its respective equilibrium position, then the potential energy stored in the rod is given by:

$$V = \frac{1}{2}\sum_i k\left(\eta_{i+1} - \eta_i\right)^2.$$

Combining our expressions for the kinetic and potential energy we see that the lagrangian for this system is

$$L = \sum_i \left[\frac{1}{2}m\left(\dot{\eta}_i\right)^2 - V\left(\eta_i\right)\right] = \sum_i \frac{1}{2}\left[m\left(\dot{\eta}_i\right)^2 - k\left(\eta_{i+1} - \eta_i\right)^2\right].$$

This can be written as

$$L = \sum_i \frac{1}{2} l \left[\frac{m}{l}\left(\dot{\eta}_i\right)^2 - kl\left(\frac{\eta_{i+1} - \eta_i}{l}\right)^2\right], \qquad (2.1)$$

where l is the lattice spacing between the masses. The idea is to construct a one dimensional scalar field out of the rod model by taking the continuum limit. That is, if the lattice spacing of the rod is initially l , then we are concerned with what happens when $l \to 0$. In this limit, the η_i 's can be replaced by a new variable $\eta\left(x,t\right)$ which may vary with respect to both position and time. For the sake of convention, $\eta\left(x,t\right)$ is written as $\varphi\left(x,t\right)$, this is called a scalar field [6, pg-17]. This subtle shift from mass positions in a lattice to a field is usually taken for granted as a simple observation, but a quick glance reveals an oddity in the transformation. Namely that, the η 's are supposed to indicate the positions of the particles, however, they transform to φ which is dependent on position. The point to be made is that the η 's indicate how stretched each spring is. In the continuum limit of this particular rod example (when the rod essentially becomes a string of rubber) φ tells us how stretched the rubber string is at any given point on the string. Returning to equation (2.1), we want to see what the Lagrangian becomes in the continuum limit, starting with the kinetic energy term

$$K = \sum_i \frac{1}{2} m \left(\dot{\eta}_i \right)^2 .$$

In the continuum limit there is a uniform mass density. Thus, the kinetic energy of a bit of mass contained by a differential line segment is

$$dK = \frac{1}{2} v^2 dm = \frac{1}{2} \rho \left(\frac{\partial \varphi}{\partial t} \right)^2 dx.$$

The total kinetic energy is then found by integrating over all space (in our case the one dimensional real line).

$$K = \int \frac{1}{2} \rho \left(\frac{\partial \varphi}{\partial t} \right)^2 dx. \tag{2.2}$$

In the case of the potential energy, we notice that

$$\frac{\eta_{i+1}-\eta_i}{l}=\frac{\eta(x+l)-\eta(x)}{l}.$$

Taking the continuum limit of the above expression we have

$$\frac{\eta_{i+1}-\eta_i}{l} \rightarrow \lim_{l\rightarrow 0}\left[\frac{\eta(x+l)-\eta(x)}{l}\right].$$

We recognize this as the definition of a partial derivative, thus

$$\frac{\eta_{i+1}-\eta_i}{l}\rightarrow\frac{\partial\eta}{\partial x}.$$

The potential energy in the continuum limit is then

$$V=\int\frac{1}{2}Y\left(\frac{\partial\eta}{\partial x}\right)^2 dx,$$

where Y is Young's Modulus. Remembering the aforementioned convention ($\eta\rightarrow\varphi$) for a scalar field the potential energy becomes

$$V=\int\frac{1}{2}Y\left(\frac{\partial\varphi}{\partial x}\right)^2 dx, \qquad (2.3)$$

where Y is kl in the continuum limit (see appendix 1). Combining equations (2.2) and (2.3) we arrive at the lagrangian of the continuous system

$$L = \int \frac{1}{2}\left[\rho\left(\frac{\partial\varphi}{\partial t}\right)^2 - Y\left(\frac{\partial\varphi}{\partial x}\right)^2\right]dx.$$

Let $Y = \rho c^2$ and scale the field such that $\varphi \to \frac{\varphi}{\sqrt{\rho}}$, then the lagrangian becomes

$$L = \int \frac{1}{2}\left[\rho\left(\frac{1}{\sqrt{\rho}}\frac{\partial\varphi}{\partial t}\right)^2 - \rho c^2\left(\frac{1}{\sqrt{\rho}}\frac{\partial\varphi}{\partial x}\right)^2\right]dx$$

$$= \int \frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial t}\right)^2 - c^2\left(\frac{\partial\varphi}{\partial x}\right)^2\right]dx. \qquad (2.4)$$

The action of the system is found by integrating (2.4) over time,

$$S(\varphi) = \int_0^T \int \frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial t}\right)^2 - c^2\left(\frac{\partial\varphi}{\partial x}\right)^2\right]dxdt. \qquad (2.5)$$

Starting with a simple classical example we have managed to create a Lorentz invariant action (see appendix 2). It is not the contemporary view that the scalar fields of physics are composed of infinitesimally small oscillators. This is a simple physical example that shows us the form in which the action should take based on classical principles. It will also give us insight later as to how to quantize the field. The modern view [6, pg-17] is to first construct an action which obeys a specific symmetry rule, in this case Lorentz invariance. So more generally, we can start with the Lorentz invariant action of the form

$$S=\int d^d x\left[\frac{1}{2}\left(\frac{\partial\varphi}{\partial t}\right)^2-\frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2\right]-\frac{1}{2}m^2\varphi^2-\frac{g}{3!}\varphi^3+.....\right]$$
$$=\int d^d x\left[\frac{1}{2}(\partial\varphi)^2-\frac{1}{2}m^2\varphi^2-\frac{g}{3!}\varphi^3+.....\right]$$

$$=\int d^d xL\left(\frac{\partial\varphi}{\partial t},\frac{\partial\varphi}{\partial x_\nu},\varphi\right), \tag{2.6}$$

where we have taken c=1, $(\partial\varphi)^2=\left(\frac{\partial\varphi}{\partial t}\right)^2-\left(\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2\right)$, and d is the dimension of the spacetime that we are concerned with, for our purposes four-dimensional Minkowski space. As with all action principles, we want to find the equations which result from extremizing the action. These are the Euler-Lagrange equations [6, pg-19] which have the general form,

$$\partial_\mu\left(\frac{\partial L}{\partial\left(\partial_\mu\varphi\right)}\right)-\frac{\partial L}{\partial\varphi}=0,$$

where Einstein summation convention is being used. For our specific lagrangian density, this becomes

$$\partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\varphi)}\right)-\frac{\partial L}{\partial\varphi}=\partial_0\left(\frac{\partial L}{\partial(\partial_0\varphi)}\right)+\partial_1\left(\frac{\partial L}{\partial(\partial_1\varphi)}\right)+\partial_2\left(\frac{\partial L}{\partial(\partial_2\varphi)}\right)+\partial_3\left(\frac{\partial L}{\partial(\partial_3\varphi)}\right)-\frac{\partial L}{\partial\varphi}=0$$

$$=\frac{\partial^2\varphi}{\partial {x_0}^2}-\left(\frac{\partial^2\varphi}{\partial {x_1}^2}+\frac{\partial^2\varphi}{\partial {x_2}^2}+\frac{\partial^2\varphi}{\partial {x_3}^2}\right)+m^2\varphi+\frac{g}{2}\varphi^2+\frac{\lambda}{6}\varphi^3+.....=0$$

$$=\left(\partial^2+m^2\right)\varphi+\frac{g}{2}\varphi^2+\frac{\lambda}{6}\varphi^3+.....=0. \qquad (2.7)$$

This is the general field equation for our system. Now that we have arrived at the field equation, it is important to reflect on why we chose a Lorentz invariant action. We notice that since we chose a Lorentz invariant action we were able to generate a field equation which takes the same form for every inertial observer. We want the laws of physics to be the same for every inertial reference frame so that there is no preferred reference frame. Otherwise, the laws of physics could change between inertial reference frames. In the classical sense this would be analogous to Newton's second law taking different forms depending on which reference frame you are in,

$$\vec{F}_{net}=m\vec{a}$$
$$\vec{F}_{net}^{'}=2m\vec{a}\,'.$$

In order to do physics, we would have to know which inertial frame we were in (as if there was a hierarchy of frames) and which law of physics corresponds to that particular reference frame. In many ways it is empirically and conceptually treacherous (if not impossible) to distinguish between inertial reference frames and the laws of physics in this fashion. So in order to do physics, we choose the laws of physics to take the same form in each inertial reference frame. Working backwards, we can achieve this by creating a Lorentz invariant action.

Now that we have constructed the classical field theory, we need to quantize the field. The first attempt at creating a relativistic equation involving a wave function [7, -pg-29] for a spinless free particle used the relativistic energy relationship:

$$E^2 = c^2 P^2 + m^2 c^4 . \tag{2.8}$$

Substituting in operators from standard quantum theory,

$$\hat{E} = i\hbar \frac{\partial}{\partial t}, \;\; \hat{P} = -i\hbar \nabla,$$

and acting them on a wave function φ , equation (2.8) becomes

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \varphi + \frac{m^2 c^2}{\hbar^2} \varphi = 0.$$

Choosing $c = \hbar = 1$, the wave equation takes the form

$$\left(\partial^2 + m^2 \right) \varphi = 0. \tag{2.9}$$

This is the Klein-Gordon equation and it was initially met with great skepticism. The first of its two main defects is that the probability density is not positive definite [7, pg-30]. In standard quantum theory, a positive definite probability density is necessary in order to make sensible physical predictions. The other problem was discovered after solving (2.9) for φ. For a real scalar field, equation (2.9) has the plane wave solution

$$\varphi(\vec{x},t) = Ae^{-i\left(\omega_k t - \vec{k}\bullet\vec{x}\right)} + A^* e^{i\left(\omega_k t - \vec{k}\bullet\vec{x}\right)}, \tag{2.10}$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$ so that equation (2.8) is satisfied (with $c = \hbar = 1$, so $E = \omega$, $\vec{k} = \vec{P}$). Since the field is free, there are no restraints on k and the general solution for the field becomes a continuous linear combination of solutions

$$\varphi(\vec{x},t) = \int\left(A\left(\vec{k}\right)e^{-i\left(\omega_k t - \vec{k}\bullet\vec{x}\right)} + A^*\left(\vec{k}\right)e^{i\left(\omega_k t - \vec{k}\bullet\vec{x}\right)}\right)d^D k\,. \tag{2.11}$$

If the two terms in equation (2.10) are considered as wave functions [13, pg-170], then the first would correspond to a particle of momentum $\vec{k}$ and positive energy E_k. The second term would represent the wave function of a particle with momentum - $\vec{k}$ and negative energy - E_k. The Klein-Gordon equation then allows for the existence of negative as well as positive energy states. This means that there is no non-arbitrary ground state and a particle could continuously drop energy states forever emitting photons. Such instability is not observed in the quantum world, so the original Klein-Gordon approach was seen as untenable.

The problem with the initial Klein-Gordon equation lay completely in its interpretation, namely that it was an equation for a single particle with φ serving as a wave function. In order to remedy this, we consider a multi- particle interpretation where φ is a field operator rather than a wave function. Taking the canonical approach we recall for classical systems

$$L = \frac{1}{2}m\left(\dot{q}\right)^2 - V\left(q\right).$$

The generalized momentum and classical Hamilton-Jacobi integral are given by,

$$p = \frac{\partial L}{\partial \dot{q}},$$
$$h = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L.$$

Substituting the generalized momentum for the $\dot{q}$'s in the Hamilton-Jacobi integral yields the Hamiltonian. Following Heisenberg, we consider p, q to be operators and impose the commutator conditions,

$$\begin{aligned}
&[p,q]=-i \\
&\frac{dp}{dt}=i[H,p]=-V(q) \\
&\frac{dq}{dt}=i[H,q]=p.
\end{aligned} \tag{2.12}$$

We can extend these concepts to field theory. Starting with the classical lagrangian embedded in equation (2.6) and dropping the higher-order terms, we have

$$L=\int d^D x\left(\frac{1}{2}\left(\left(\frac{\partial\varphi}{\partial t}\right)^2-(\nabla\varphi)^2-m^2\varphi^2\right)\right),$$

where D is the spatial dimension of our spacetime. Then the conjugate momentum density to the field is

$$\pi(\vec{x},t)=\frac{\partial L}{\partial\left(\frac{\partial\varphi}{\partial t}\right)}=\frac{\partial\varphi(\vec{x},t)}{\partial t}.$$

Analogous to the canonical commutation relationship between momentum and position, we have

$$\begin{aligned}
&\left[\pi(\vec{x},t),\vec{\varphi}(\vec{x}',t)\right]=\left[\frac{\partial\varphi(\vec{x},t)}{\partial t},\vec{\varphi}(\vec{x}',t)\right]=-i\delta^{(D)}(\vec{x}-\vec{x}'), \\
&\left[\vec{\varphi}(\vec{x},t),\vec{\varphi}(\vec{x}',t)\right]=\left[\pi(\vec{x},t),\pi(\vec{x}',t)\right]=0.
\end{aligned} \tag{2.13}$$

Following the classical construction of the Hamiltonian from the Hamilton-Jacobi integral we can determine the quantum field Hamiltonian [7, pg-130],

$$H = \int d^D x \left[\frac{L}{\partial\left(\partial_\mu \varphi\right)} \bullet \partial_\upsilon \varphi - \delta_\upsilon^\mu L \right]$$
$$= \int d^D x \left(\frac{1}{2} \left[\pi^2 + \nabla\varphi \bullet \nabla\varphi + m^2 \varphi^2 \right] \right)$$

$$= \int d^D x \left(\frac{1}{2} \left[\left(\partial_0 \varphi\right)^2 + \nabla\varphi \bullet \nabla\varphi + m^2 \varphi^2 \right] \right). \quad (2.14)$$

The Hamiltonian here is positive definite [7, pg-130], thus the scalar field is not ruined by a negative energy problem. Now we must further investigate the field operator φ. In standard quantum mechanical fashion we expect the field operator to be Hermitian, whose Fourier expansion can be written as:

$$\varphi(\vec{x},t) = \int \frac{1}{\sqrt{(2\pi)^D 2\omega_k}} \left(a\left(\vec{k}\right) e^{-i\left(\omega_k t - \vec{k} \bullet \vec{x}\right)} + a^\dagger\left(\vec{k}\right) e^{i\left(\omega_k t - \vec{k} \bullet \vec{x}\right)} \right) d^D k\,, \quad (2.15)$$

where again $\omega_k = \sqrt{\vec{k}^2 + m^2}$ and $a\left(\vec{k}\right)$, $a^\dagger\left(\vec{k}\right)$ are operators. Without loss of generality we consider the one dimensional case and the operators $a(k)$, $a^\dagger(k)$. The commutators of these new operators are [7, pg 132]:

$$
\begin{aligned}
\left[a(k),a(k')\right]&=\left[a^\dagger(k),a^\dagger(k')\right]=0\\
\left[a(k),a^\dagger(k')\right]&=\delta(k-k').
\end{aligned}
\tag{2.16}
$$

To determine the nature of these operators we define a new operator,

$$
N(k)=a^\dagger(k)a(k). \tag{2.17}
$$

Let the eigenvalue of this operator be $n(k)$ such that

$$
N(k)\left|n(k)\right\rangle=n(k)\left|n(k)\right\rangle. \tag{2.18}
$$

Now we determine the commutation relations,

$$
\begin{aligned}
\left[N(k),a^\dagger(k)\right]&=N(k)a^\dagger(k)-a^\dagger(k)N(k)\\
&=a^\dagger(k)a(k)a^\dagger(k)-a^\dagger(k)a^\dagger(k)a(k)\\
&=a^\dagger(k)\left[a(k),a^\dagger(k)\right]\\
&=a^\dagger(k).
\end{aligned}
\tag{2.19}
$$

Similarly,

$$
\left[N(k),a(k)\right]=N(k)a(k)-a(k)N(k)=-a(k)\ . \tag{2.20}
$$

From equation (2.19) we see that

$$
\begin{aligned}
N(k)a^\dagger(k)|n(k)\rangle &= a^\dagger(k)N(k)|n(k)\rangle + a^\dagger(k)|n(k)\rangle \\
&= a^\dagger(k)n(k)|n(k)\rangle + a^\dagger(k)|n(k)\rangle \\
&= (n(k)+1)a^\dagger(k)|n(k)\rangle.
\end{aligned}
\tag{2.21}
$$

Similarly from equation (2.20) we get:

$$
N(k)a(k)|n(k)\rangle = (n(k)-1)a(k)|n(k)\rangle. \tag{2.22}
$$

Comparing (2.21), (2.22) to (2.18) we see that the operators $a(k)$, $a^\dagger(k)$ must behave according to the following rules,

$$
\begin{aligned}
a^\dagger(k)|n(k)\rangle &= |n(k)+1\rangle \\
a(k)|n(k)\rangle &= |n(k)-1\rangle.
\end{aligned}
\tag{2.23}
$$

This suggests that the operator $N(k)$ is the operator for the number of particles with momentum k and $n(k)$ is then the number of particles with momentum k, also known as the occupation number. $a^\dagger(k)$ and $a(k)$ are thus called creation and annihilation operators because they literally create and remove particles. This is completely analogous to the raising and lowering operators of the quantum mechanical oscillator. We should also remember from the quantum oscillator that in order to assure normalization our new operators should take the form:

$$a^\dagger(k)\left|n(k)\right\rangle = \sqrt{n(k)+1}\left|n(k)+1\right\rangle$$
$$a(k)\left|n(k)\right\rangle = \sqrt{n(k)}\left|n(k)-1\right\rangle. \quad (2.24)$$

For a system where possible momentum values are finite, the momentum state takes the form,

$$\left|n(k_i)\right\rangle = \left|n(k_1)n(k_2)n(k_3).....\right\rangle.$$

We define the ground state to be

$$\left|0\right\rangle = \left|0,0,0,.....,0\right\rangle$$
$$a(k)\left|0\right\rangle = 0 \quad (2.25)$$

We notice that two creation operators acting on a state can create two particles with the same momentum. This indicates that the Klein-Gordon field is a bosonic field. To deal with fermions the raising and lowering operators must be altered such that no particles are produced in the same state. This also amounts to changing the field equation, but for our purposes the bosonic Klein-Gordon field is sufficient.

The Klein-Gordon field is a great example of a quantized field. We can extend quantum field theory to the curved spacetime of general relativity by utilizing the action principle developed in the beginning of this section. It is important to note that the Klein-Gordon equation (2.9) can be derived from the Euler-Lagrange equations associated with an action principle. This can be seen in equation (2.7), which reduces to (2.9) under the free field assumption (second order and higher terms in the field are neglected). The motivation for creating an action of the form (2.6) was to ensure that the action takes the same form under Lorentz coordinate transformations. In general relativity each observer has an associated coordinate frame which may or may not be linked to other observers via Lorentz transformations. Considering the broad and varying use of coordinate systems in general relativity we want to create an action principle that is invariant under coordinate transformations. Since the spacetime structure is defined by the metric tensor we expect the action to dependent on both metric components as well as field components. Following the form of (2.6) we choose

$$S = \int dx^4 \sqrt{-g}\,\frac{1}{2}\left(g^{\mu\upsilon}\partial_\mu\varphi\partial_\upsilon\varphi - m^2\varphi^2\right), \tag{2.26}$$

where g is the determinant of the metric. This is a coordinate-invariant action (see appendix 3) for a free field in curved spacetime. It is observed that the free field form of equation (2.6) is retrieved when the arbitrary metric g becomes the Minkowski metric η . If we consider a massless scalar field in a 1+1 dimensional spacetime, then the action is invariant under conformal transformations of the form,

$$g_{\mu\upsilon} \to g'_{\mu\upsilon} = \psi^2(x)\,g_{\mu\upsilon} \ , \tag{2.27}$$

where $\psi^2(x)$ is an arbitrary function. Since the determinant is only 2x2, the determinant of the metric has powers of ψ^4. The square root thus yields a second order factor in ψ which cancels with the inverse metric transformation,

$$g^{\mu\upsilon} \to g'^{\mu\upsilon} = \frac{1}{\psi^2(x)} g^{\mu\upsilon}.$$

The massless action is therefore invariant under such a transformation.

3.Unruh Effect

The Unruh effect occurs when an accelerated observer sees black body radiation in a region of spacetime where an inertial observer does not. In the previous section we have seen how the particle nature of the field is contained in equation (2.15). For an inertial observer, (2.15) is defined with respect to the coordinate time t, while for the accelerated observer (2.15) is naturally defined with respect to proper time τ. Because of this difference in construction, what is registered as a particle for one observer will not necessarily register as a particle for the other. The accelerated observer will find the vacuum as a state containing particles and with an associated temperature. This section follows [8].

Consider an observer undergoing uniform one dimensional acceleration along the x -axis. The observer's trajectory according to the inertial observer is given by (ref. appendix.4)

$$x(\tau)=\frac{1}{|a|}\cosh(|a|\tau), \quad t(\tau)=\frac{1}{|a|}\sinh(|a|\tau). \tag{3.1}$$

Using the light cone coordinates:

$$u=t-x, \quad v=t+x, \tag{3.2}$$

the Minkowski metric can be written as,

$$ds^2 = dudv\,. \tag{3.3}$$

The trajectories of equation (3.1) become

$$u(\tau) = -\frac{1}{|a|}e^{-|a|\tau}, \quad v(\tau) = \frac{1}{|a|}e^{|a|\tau}\,. \tag{3.4}$$

For a comoving frame of the accelerated observer we introduce the coordinates $\left(\xi^0, \xi^1\right)$.

According to the comoving frame, the accelerated observer is at rest at $\xi^1 = 0$ and ξ^0 coincides with the proper time along the observer's world line. We define the accelerated observer's coordinates away from his world line by imposing the condition that the metric take the following form

$$ds^2 = \psi^2\left(\xi^0, \xi^1\right)\left[\left(d\xi^0\right)^2 - \left(d\xi^1\right)^2\right], \tag{3.5}$$

where the term in brackets has the Minkowski signature (+,-), and $\psi^2\left(\xi^0, \xi^1\right)$ is to be determined. The lightcone coordinates for the accelerated frame are

$$\bar{u} = \xi^0 - \xi^1, \quad \bar{v} = \xi^0 + \xi^1\,. \tag{3.6}$$

Under this coordinate transformation (3.5) becomes

$$ds^2 = \psi^2(\bar{u},\bar{v})\, d\bar{u}d\bar{v} \,. \tag{3.7}$$

Again the accelerated observers' world line is

$$\xi^0(\tau) = \tau, \quad \xi^1(\tau) = 0 \,. \tag{3.8}$$

Or in lightcone coordinates the world line is:

$$\bar{u}(\tau) = \bar{v}(\tau) = \tau \,. \tag{3.9}$$

In order for the metric to be conformal about the observer's world line we see that,

$$\psi^2(\bar{u},\bar{v})\Big|_{\bar{u}=\bar{v}=\tau} = 1, \tag{3.10}$$

since ξ^0 is the proper time at the accelerated observer's world line. The metric according to the inertial observer's coordinates and the metric according to the accelerated observer's coordinates are the same considering that they both describe the same Minkowski spacetime, thus

$$ds^2 = dudv = \psi^2(\bar{u},\bar{v})\, d\bar{u}d\bar{v} \,. \tag{3.11}$$

With regards to the coordinate transformation between (u,v) and $(\bar{u},\bar{v})$ we choose the coordinate u to be dependent only on $\bar{u}$ and the coordinate v to be dependent only on $\bar{v}$. This is done in order to preserve the form of equation (3.7), so that no second order terms appear in the metric. By the chain rule we have

$$\frac{du(\tau)}{d\tau}=\frac{du(\bar{u})}{d\bar{u}}\frac{d\bar{u}(\tau)}{d\tau} . \tag{3.12}$$

Recalling equations (3.9) and (3.4), equation (3.12) becomes

$$\frac{du(\bar{u})}{d\bar{u}}=-|a|u ,$$

which has a solution of the form

$$u=Ae^{-|a|\bar{u}} . \tag{3.13}$$

Similarly for v we have

$$v=Be^{|a|\bar{v}} . \tag{3.14}$$

Equation (3.10) restricts the integration constants A, B to satisfy

$$|a|^2 AB = -1.$$

Choosing $A = -B$, equations (3.13) and (3.14) become

$$u = \frac{-1}{|a|} e^{-|a|\bar{u}}, \quad v = \frac{1}{|a|} e^{|a|\bar{v}}, \tag{3.15}$$

and the metric is

$$ds^2 = e^{|a|(\bar{v}-\bar{u})} d\bar{u} d\bar{v}. \tag{3.16}$$

Combining equations (3.16) and (3.6) we arrive at a commonly used version of the metric

$$ds^2 = e^{2|a|\xi^1} \left[\left(\xi^0\right)^2 - \left(\xi^1\right)^2 \right]. \tag{3.17}$$

This metric is locally equivalent to Minkowski spacetime and describes Rindler spacetime.

We have determined the effective coordinates (see appendix 7) for the inertial and accelerated observer. Now we can analyze how quantum field theory behaves in each reference frame. For a massless scalar field we use the action developed at the end of section 2,

$$S = \int dx^4 \sqrt{-g}\,\frac{1}{2}\left(g^{\mu\upsilon}\partial_\mu\varphi\partial_\upsilon\varphi\right). \qquad (3.18)$$

The action for the inertial observer is

$$S = \frac{1}{2}\int\left[\left(\partial_t\varphi\right)^2 - \left(\partial_x\varphi\right)^2\right]dtdx. \qquad (3.20)$$

From equation (3.5) we see that the accelerated observer's metric has a conformal factor of ψ^2, and so his action takes the form

$$S = \frac{1}{2}\int\left[\left(\partial_{\xi^0}\varphi\right)^2 - \left(\partial_{\xi^1}\varphi\right)^2\right]d\xi^0 d\xi^1. \qquad (3.21)$$

Using the Euler-Lagrange equations (2.7), we have for the inertial observer,

$$\partial_\mu\left(\frac{\partial L}{\partial\left(\partial_\mu\varphi\right)}\right) - \frac{\partial L}{\partial\varphi} = \partial_t\left(\frac{\partial}{\partial\left(\partial_t\varphi\right)}\left[\left(\partial_t\varphi\right)^2 - \left(\partial_x\varphi\right)^2\right]\right) + \partial_x\left(\frac{\partial}{\partial\left(\partial_x\varphi\right)}\left[\left(\partial_t\varphi\right)^2 - \left(\partial_x\varphi\right)^2\right]\right) = 0$$

$$= \frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} = 0\,.$$

The field equation for the Rindler observer is identical in form. The field equations have plane wave solutions. Following the quantization of the Klein-Gordon field in section (2), this field is given by

$$\begin{aligned}\varphi &= \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}}\left[a(\omega)e^{-i\omega(t-x)} + a^\dagger(\omega)e^{i\omega(t-x)}\right] \\ &= \int_0^\infty \frac{d\Omega}{\sqrt{4\pi\Omega}}\left[b(\Omega)e^{-i\Omega(\xi^0-\xi^1)} + b^\dagger(\Omega)e^{i\Omega(\xi^0-\xi^1)}\right].\end{aligned} \tag{3.22}$$

The Minkowski vacuum is then defined by

$$a(\omega)|0_M\rangle = 0,$$

while the Rindler vacuum is defined by

$$b(\Omega)|0_R\rangle = 0.$$

The Rindler lowering operator is linked to the Minkowski operators via the Bogoliubov transformations [11, pg-106],

$$\begin{aligned}b(\Omega) &= \int_0^\infty d\omega\left[\alpha_{\Omega\omega}a(\omega) - \beta_{\Omega\omega}a^\dagger(\omega)\right] \\ b^\dagger(\Omega) &= \int_0^\infty d\omega\left[\alpha^*_{\Omega\omega}a^\dagger(\omega) - \beta^*_{\Omega\omega}a(\omega)\right].\end{aligned} \tag{3.23}$$

Substituting (3.23) into (3.22) and converting to light cone coordinates we get on the right side of the equation

$$\varphi = \int_0^\infty \frac{d\Omega}{\sqrt{4\pi\Omega}}\left[b(\Omega)e^{-i\Omega\bar{u}} + b^\dagger(\Omega)e^{i\Omega\bar{u}}\right]$$
$$= \int_0^\infty \frac{d\Omega}{\sqrt{4\pi\Omega}}\left[\int_0^\infty d\omega\left[\alpha_{\Omega\omega}a(\omega) - \beta_{\Omega\omega}a^\dagger(\omega)\right]e^{-i\Omega\bar{u}} + \int_0^\infty d\omega\left[\alpha^*{}_{\Omega\omega}a^\dagger(\omega) - \beta^*{}_{\Omega\omega}a(\omega)\right]e^{i\Omega\bar{u}}\right]$$
$$= \int_0^\infty \frac{d\Omega}{\sqrt{4\pi\Omega}}\left[\int_0^\infty d\omega\left[\alpha_{\Omega\omega}a(\omega)e^{-i\Omega\bar{u}} - \beta^*{}_{\Omega\omega}a(\omega)e^{i\Omega\bar{u}}\right] + \int_0^\infty d\omega\left[\alpha^*{}_{\Omega\omega}a^\dagger(\omega)e^{i\Omega\bar{u}} - \beta_{\Omega\omega}a^\dagger(\omega)e^{-i\Omega\bar{u}}\right]\right]$$
$$= \int_0^\infty \frac{d\Omega}{\sqrt{4\pi\Omega}}\left[\int_0^\infty d\omega a(\omega)\left[\alpha_{\Omega\omega}e^{-i\Omega\bar{u}} - \beta^*{}_{\Omega\omega}e^{i\Omega\bar{u}}\right] + \int_0^\infty d\omega a^\dagger(\omega)\left[\alpha^*{}_{\Omega\omega}e^{i\Omega\bar{u}} - \beta_{\Omega\omega}e^{-i\Omega\bar{u}}\right]\right]$$

$$\begin{aligned} &= \int_0^\infty d\omega a(\omega)\left[\int_0^\infty \frac{d\Omega}{\sqrt{4\pi\Omega}}\left[\alpha_{\Omega\omega}e^{-i\Omega\bar{u}} - \beta^*{}_{\Omega\omega}e^{i\Omega\bar{u}}\right]\right] \\ &\quad + \int_0^\infty d\omega a^\dagger(\omega)\left[\int_0^\infty \frac{d\Omega}{\sqrt{4\pi\Omega}}\left[\alpha^*{}_{\Omega\omega}e^{i\Omega\bar{u}} - \beta_{\Omega\omega}e^{-i\Omega\bar{u}}\right]\right]. \end{aligned} \tag{3.24}$$

Compare this to the top term in equation (3.22). By equating terms that have a common operator we get

$$\frac{1}{\sqrt{\omega}}e^{-i\omega u} = \int_0^\infty \frac{d\Omega}{\sqrt{\Omega}}\left[\alpha_{\Omega\omega}e^{-i\Omega\bar{u}} - \beta^*{}_{\Omega\omega}e^{i\Omega\bar{u}}\right]. \tag{3.25}$$

We can use the orthogonality property of exponential functions to determine the coefficients α, β. Multiplying by $\exp\left(\pm i\Omega'\bar{u}\right)$ and then integrating over all space yields

$$\alpha_{\Omega'\omega} = \frac{1}{2\pi}\sqrt{\frac{\Omega'}{\omega}}\int_{-\infty}^{\infty} e^{-i\omega u + i\Omega'\bar{u}}\, d\bar{u}$$

$$\beta_{\Omega'\omega} = \frac{1}{2\pi}\sqrt{\frac{\Omega'}{\omega}}\int_{-\infty}^{\infty} e^{i\omega u + i\Omega'\bar{u}}\, d\bar{u}\ . \qquad (3.26)$$

Using equation (3.15) we can express (3.26) completely in terms of u .

$$\alpha_{\Omega'\omega} = \frac{1}{2\pi}\sqrt{\frac{\Omega'}{\omega}}\int_{-\infty}^{0} \left(-|a|u\right)^{-\frac{i\Omega'}{|a|}-1} e^{-i\omega u}\, du$$

$$\beta_{\Omega'\omega} = \frac{1}{2\pi}\sqrt{\frac{\Omega'}{\omega}}\int_{-\infty}^{0} \left(-|a|u\right)^{-\frac{i\Omega'}{|a|}-1} e^{+i\omega u}\, du\ . \qquad (3.27)$$

The integral can be calculated yielding

$$\alpha_{\Omega'\omega} = \frac{1}{2\pi|a|}\sqrt{\frac{\Omega'}{\omega}}\, e^{\left(\frac{\pi\Omega'}{2|a|}\right)} e^{\left(\frac{i\Omega'}{|a|}\ln\left(\frac{\omega}{|a|}\right)\right)}\, \Gamma\left(-\frac{i\Omega'}{|a|}\right)$$

$$\beta_{\Omega'\omega} = -\frac{1}{2\pi|a|}\sqrt{\frac{\Omega'}{\omega}}\, e^{-\left(\frac{\pi\Omega'}{2|a|}\right)} e^{\left(\frac{i\Omega'}{|a|}\ln\left(\frac{\omega}{|a|}\right)\right)}\, \Gamma\left(-\frac{i\Omega'}{|a|}\right). \qquad (3.28)$$

Then the coefficients obey the following relationship

$$\left|\alpha_{\Omega'\omega}\right|^2 = e^{\left(\frac{2\pi\Omega'}{|a|}\right)} \left|\beta_{\Omega'\omega}\right|^2. \qquad (3.29)$$

The raising and lowering operators define particles for each observer. Now we compute the expectation value of the number operator for the Rindler observer in the Minkowski vacuum. Using equation (2.17) we have

$$\langle N_\Omega \rangle = \left\langle 0_M \left| b^\dagger(\Omega) b(\Omega) \right| 0_M \right\rangle .$$

Next we substitute in the Bogoliubov transformations (3.23)

$$\langle N_\Omega \rangle = \left\langle 0_M \left| \int_0^\infty d\omega \left[\alpha^*{}_{\omega\Omega} a^\dagger(\omega) - \beta^*{}_{\omega\Omega} a(\omega) \right] \int_0^\infty d\omega' \left[\alpha_{\omega'\Omega} a(\omega') - \beta_{\omega'\Omega} a^\dagger(\omega') \right] \right| 0_M \right\rangle$$

$$= \int_0^\infty d\omega \left| \beta_{\omega\Omega} \right|^2 . \qquad (3.30)$$

This is the mean number of particles with frequency Ω detected by the accelerated observer [8, pg-107]. The normalization condition for the α, β coefficients (ref. appendix 4) is

$$\int_0^\infty d\omega \left(\left| \alpha_{\Omega\omega} \right|^2 - \left| \beta_{\Omega\omega} \right|^2 \right) = \delta(\Omega - \Omega'). \qquad (3.31)$$

Substituting (3.29) into (3.31) we get

$$\delta(\Omega-\Omega') = \int_0^\infty d\omega \left(e^{\left(\frac{2\pi\Omega}{|a|}\right)} |\beta_{\Omega\omega}|^2 - |\beta_{\Omega\omega}|^2 \right)$$
$$= \left(e^{\left(\frac{2\pi\Omega}{|a|}\right)} - 1 \right) \int_0^\infty d\omega |\beta_{\Omega\omega}|^2 . \qquad (3.32)$$

Then substituting in (3.30) we get

$$\langle N_\Omega \rangle = \delta(\Omega-\Omega') \left(e^{\left(\frac{2\pi\Omega}{|a|}\right)} - 1 \right)^{-1} . \qquad (3.33)$$

The thermal Boson-Einstein distribution [9] for the expectation value of the number operator is proportional to

$$= \delta(\Omega-\Omega') \left(e^{\left(\frac{\hbar\Omega}{kT}\right)} - 1 \right)^{-1} , \qquad (3.34)$$

where T is the temperature and k is the Boltzmann constant. Comparing equations (3.34) and (3.33) we have

$$T = \frac{|a|\hbar}{2\pi k} .$$

We notice that the units are not correct. This is because we set $c = 1$, inserting the speed of light gives the correct expression for the temperature

$$T = \frac{|a|\hbar}{2\pi kc}. \tag{3.35}$$

The accelerated observer experiences a thermal bath of temperature T proportional to his acceleration. This is an odd result considering that the accelerated observer is moving through empty space. A typical heuristic explanation is that quantum mechanics allows the vacuum to be littered with virtual particles. These particles cannot supply their own energy to excite a detector, but the energy introduced via the acceleration of the observer allows the particles to become real and detectable. This excitation of the vacuum is analogous to the Schwinger effect, where an intense electric field is used to create real particles. It should also be noted that in order to induce a sizeable temperature change an enormous acceleration is required.

4. Hawking Radiation

When quantum field theory is used in the curved spacetime around black holes it is found that black holes produce a thermal distribution of particles [3]. This will be shown for a 1+1 dimensional static black hole (this section follows [8]). A 3+1 dimensional black hole is described by the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2\left(d\theta^2 + d\phi^2 \sin^2\theta\right), \tag{4.1}$$

where $G = c = \hbar = 1$. For simplicity let $2M = m$, then the two dimensional Schwarzschild metric is given by

$$ds^2 = \left(1 - \frac{m}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{m}{r}}. \tag{4.2}$$

Next we make the coordinate transformation

$$\begin{aligned} r'(r) &= r - m + m\ln\left(\frac{r}{m} - 1\right) \\ dr' &= \frac{dr}{1 - \frac{m}{r}}. \end{aligned} \tag{4.3}$$

The metric then becomes

$$ds^2 = \left(1 - \frac{m}{r(r')}\right)\left[dt^2 - dr'^2\right]. \tag{4.4}$$

Notice that r' is only defined for $r > m$. Transforming to the modified lightcone coordinates

$$\bar{u} = t - r', \quad \bar{v} = t + r', \tag{4.5}$$

equation (4.4) becomes further simplified

$$ds^2 = \left(1 - \frac{m}{r(\bar{u},\bar{v})}\right)d\bar{u}d\bar{v}. \tag{4.6}$$

There is an apparent singularity at $r = m$ in the Schwarzschild metric. This is also true for the modified lightcone coordinates, thus they only hold good for describing the exterior of the black hole. In order to describe the whole spacetime we need to modify our coordinates. This problem is solved by introducing the Kruskal-Szekeres lightcone coordinates

$$u = -2me^{\left(-\frac{\bar{u}}{2m}\right)}, \quad v = -2me^{\left(\frac{\bar{v}}{2m}\right)}. \tag{4.7}$$

With these coordinates the metric looks like

$$ds^2 = \frac{m}{r(u,v)} e^{\left(1-\frac{r(u,v)}{m}\right)} du dv . \tag{4.8}$$

Then r is not restricted to the interior of the black hole and the whole spacetime is covered. We can make a final coordinate substitution

$$u = T - R, \quad v = T + R , \tag{4.9}$$

to get the metric into (+ -) form,

$$ds^2 = \frac{m}{r(T,R)} e^{\left(1-\frac{r(T,R)}{m}\right)} \left(dT^2 - dR^2\right). \tag{4.10}$$

The factor in front of the area element is a conformal factor and so disappears in the action. The action in equation (2.2.6) is then given by

$$S = \frac{1}{2}\int \left[\left(\partial_T \varphi\right)^2 - \left(\partial_R \varphi\right)^2 \right] dT dR . \tag{4.11}$$

We saw in the last section that actions of this form have the following field solutions:

$$\varphi = \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}}\left[a(\omega)e^{-i\omega(T-R)} + a^\dagger(\omega)e^{i\omega(T-R)}\right]$$
$$= \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}}\left[a(\omega)e^{-i\omega(u)} + a^\dagger(\omega)e^{i\omega(u)}\right]. \quad (4.12)$$

Kruskal-Szekeres light coordinates are non-singular at $r = m$ and serve as effective coordinates for an observer near the Schwarzschild horizon. Moreover, near the horizon $ds^2 \to dudv = dT^2 - dR^2$. This indicates that the Kruskal-Szekeres coordinates are a natural coordinate choice for a field observer near the horizon. Similar to what was done with the Unruh effect, we would like to compare field observations near the horizon to field observations made very far from the horizon where the spacetime is Minkowskian [8]. If we choose the modified light cone coordinates for distances very far from the horizon then, $ds^2 \to d\bar{u}d\bar{v} = dt^2 - dr'^2$. Thus, the modified light cone coordinates are a natural choice for an observer far away from the black hole. The field expansion for the observer is

$$\varphi = \int_0^\infty \frac{d\Omega}{\sqrt{4\pi\Omega}}\left[b(\Omega)e^{-i\Omega(t-r')} + b^\dagger(\Omega)e^{i\Omega(t-r')}\right]$$
$$= \int_0^\infty \frac{d\Omega}{\sqrt{4\pi\Omega}}\left[b(\Omega)e^{-i\Omega(\bar{u})} + b^\dagger(\Omega)e^{i\Omega(\bar{u})}\right]. \quad (4.13)$$

The field expansions which we wish to compare are identical in appearance to those which we have compared in the last section. Following the exact same mathematics in section (3) we would get an identical result, except, the constant acceleration would be replaced with a new constant. To see what this value is we only need to compare the similar coordinate transformations provided in equations (3.15) and (4.7),

$$u = \frac{-1}{|a|} e^{-|a|\bar{u}} \quad and \quad u = -2me^{\left(-\frac{\bar{u}}{2m}\right)}$$

$$|a| \to \frac{1}{2m} = \frac{1}{4M} . \tag{4.14}$$

Substituting this into equation (3.35) we get

$$T = \frac{\hbar}{8M\pi kc} .$$

Again it is seen that the units do not match. This is because in equation (4.1) we let $G = c = 1$. Substituting accordingly into the above expression we get

$$T = \frac{\hbar c^3}{8\pi GMk} . \tag{4.15}$$

Substituting this into equation (4.15) the expressions for the Newtonian acceleration and the Schwarzschild radius we have

$$T = \frac{\hbar g}{2\pi kc} . \tag{4.16}$$

This is exactly the same result which was obtained for the accelerated reference frame with surface gravity replacing acceleration. A result which might have been expected, considering the equivalence principle of general relativity.

We can obtain another interesting result by starting with the area of the Schwarzschild horizon

$$A = 4\pi r_s^{\,2},$$

where r_s is the Schwarzschild radius given by

$$r_s = \frac{2GM}{c^2}$$
$$\rightarrow A = \frac{16\pi G^2 M^2}{c^4}.$$

Differentiating both sides we have

$$dA = \frac{32\pi G^2 M}{c^4} dM$$
$$\rightarrow dM = \frac{c^4}{32\pi G^2 M} dA.$$

Substituting in equation (4.15) leads to

$$
\begin{aligned}
dM &= T\frac{ck}{4G\hbar}dA \\
&= Td\left(\frac{ck}{4G\hbar}A\right).
\end{aligned}
\tag{4.17}
$$

Substituting equation (1.1) into (4.17) we get

$$dE = Td\left(\frac{c^3k}{4G\hbar}A\right). \tag{4.18}$$

If we consider the change in mass energy as a form of heat transfer and T to be the temperature associated with the black hole, then (4.18) resembles the first law of thermodynamics:

$$dQ = TdS\ .$$

Thus, the Schwarzschild black hole has the following entropy

$$S = \frac{c^3k}{4G\hbar}A. \tag{4.19}$$

As was mentioned in the introductory section, black holes are believed to be active thermal bodies with the thermodynamic properties given equations (4.19) and (4.15). Black holes are also intrinsically general relativistic and so black hole thermodynamics represents at least one link between thermodynamics and general relativity. It is important to note that the derivations of the Hawking radiation were done in a 1+1 dimensional spacetime. Generalizations to the actual 3+1 dimensional spacetime introduce complications in the mathematics. For instance, the coordinate transformations that were used become more complex and in the case of Hawking radiation the field is expanded in terms of spherical harmonics [8, pg-119]. Moreover, the conformal invariance expressed in equation (2.27) is lost so that evaluating the action principle near the horizon becomes more complex. Calculations still result in the same Hawking temperature distribution which is modified by a greybody factor [3].

5. The State Equation of Gravity

This section follows Ted Jacobson's derivation of the Einstein field equations [4]. The equivalence principle allows us to consider a local portion of curved spacetime as locally flat. Let us consider an accelerated observer p in this frame. There is a spacelike area element which is perpendicular to the world line of p that has vanishing expansion θ and shear σ at the point p_0 and is accelerating with p. In accelerated reference frames, such as the ones mentioned in section (3), causal horizons form. Beyond these horizons any emitted light signal will not reach an observer maintaining constant acceleration. For the case of our locally flat frame of reference the past horizon of p_0 will be called the "local Rindler horizon". Let χ represent the vector field tangent to the observers' world line and K be the four vector lying in the direction tangent to the horizon H. We want to know the heat and entropy flow across the Rindler horizon so that we can make a thermodynamic argument similar to the one we made for the Hawking radiation. In order for our observer to get close to the horizon his acceleration must approach infinity. In this limit χ essentially points along K, and so $\chi^a = -|a|\lambda K^a$, where λ is the curve parameter that vanishes at p_0, and $|a|$ is the acceleration of χ. If we consider the flow of energy across the Rindler horizon to be a form of heat transfer then the variation of heat caused by energy flow across a differential portion of the horizon dA is given by

$$\begin{aligned} \delta Q &= \int_H T_{ab}\chi^b d\Sigma^a \\ &= -\int_H |a|\lambda T_{ab}K^a K^b d\lambda dA, \end{aligned} \tag{5.1}$$

where T_{ab} is the stress energy tensor. At the end of section (4) we discovered that the entropy of a black hole is proportional to the area of its event horizon. Following this we assume that the variation of the entropy associated with the Rindler horizon is proportional to the variation in its area

$$\delta S = \eta \delta A. \tag{5.2}$$

Since the area element is traveling through curved space time it may expand and so δA represents the variation in area as the observer approaches p_0. Thus δA is related to the expansion θ by

$$\delta A = \int_H \theta(\lambda) d\lambda dA. \tag{5.3}$$

On the differential area element we can imagine normal vectors pointing away from its surface. As the area element moves through curved space time it may warp and the normal vectors may deviate and spread apart from each other. This deviation is related to how curved the spacetime is. The deviation is specifically expressed in the Raychaudhuri equation [10, pg-222] for null geodesic deviation

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^2 - R_{ab}K^a K^b. \tag{5.4}$$

We consider null geodesic deviation because the acceleration of the area element is such that its world line is approximately light-like. Since θ and σ under our construction are very small, their squares are negligible. This leaves

$$\frac{d\theta}{d\lambda} = -R_{ab}k^a k^b.$$

The right side does not vary with respect to λ so we get

$$\theta(\lambda) = -\lambda R_{ab}K^a K^b.$$

Substituting this result into equation (5.3) gives

$$\delta A = -\int_H \lambda R_{ab}K^a K^b d\lambda dA, \tag{5.5}$$

and the variation in entropy is

$$\delta S = -\eta \int_H \lambda R_{ab}K^a K^b d\lambda dA. \tag{5.6}$$

Considering that an accelerated observer riding along with the area element experiences a temperature according to the Unruh effect, and using this temperature, equation (5.6), and equation (5.1), the first law of thermodynamics gives

$$\delta Q = T\delta S$$

$$-\int_H |a| \lambda T_{ab} K^a K^b d\lambda dA = -\frac{|a|}{2\pi} \eta \int_H \lambda R_{ab} K^a K^b d\lambda dA, \tag{5.7}$$

where we have taken the fundamental constants to be unity. Note that the first law of thermodynamics also includes a work term which is not present in equation (5.7) because no work is being done by or on the system. We are simply looking at the heat flow across a differential portion of the horizon. Equation (5.7) can only hold if

$$T_{ab} K^a K^b = \frac{1}{2\pi} \eta R_{ab} K^a K^b. \tag{5.8}$$

This is true for an arbitrary null vector K, thus we have

$$\frac{2\pi}{\eta} T_{ab} = R_{ab} + f g_{ab}, \tag{5.9}$$

where f is an undetermined function and g_{ab} is the metric tensor. Local conservation of energy requires that the stress energy tensor be divergence free. This is satisfied by the Bianchi identity with $f = \left(\frac{-R}{2} + \Lambda \right)$, where R is the scalar curvature (ref. appendix.6). Equation (5.9) becomes

$$\frac{2\pi}{\eta} T_{ab} = R_{ab} - \frac{R}{2} g_{ab} + \Lambda g_{ab}. \tag{5.10}$$

We recognize this as the Einstein field equations for general relativity with Λ serving as the cosmological constant. By imposing the first law of thermodynamics we were able to derive the Einstein field equations. In general we can insist that any law of gravity be consistent with thermodynamics. This provides a means for checking viable new theories.

6. Modified Gravity

In spiral galaxies we expect that each star travels in a circular path about the galactic center. The gravitational strength outside the center is sufficiently weak and Newton's theory of gravitation is a good approximation for stellar dynamics. If we consider the galactic center to be far more massive then the surrounding stars then we can approximate it as a single massive body. To determine the velocity of a star outside the galactic center we use Newton's second law:

$$\begin{aligned} \vec{F} &= m\vec{a} \\ \frac{GMm}{r^2} &= m\frac{v^2}{r} \\ \rightarrow v &= \sqrt{\frac{GM}{r}}\,, \end{aligned} \tag{6.1}$$

where r is the radial distance from the galactic center to a particular star we are looking at. From this we expect that stars which are farther from the center travel more slowly than stars nearer to the center. This expectation parallels the velocity distribution of the planets in our solar system about the sun. However, experimental evidence [12, 13, 14] indicates that galactic star velocity is constant with respect to changes in radial distance. A possible explanation for this mystery is that the total mass distribution of the galaxy has not been accounted for [14]. With the correct additional mass distribution the observed velocities make sense. However, this additional mass must be very hard to detect. In fact, it must not reflect or emit light otherwise we would have observed it. This is "dark matter," and it is not only difficult to detect but, it is also estimated that it accounts for the majority of the matter in our galaxy. So far, not much experimental evidence has surfaced that verifies the existence of dark matter. This has led some to appeal to an alternative theory of gravity rather than dark matter. The simplest way to start is to modify Newton's law of gravitation. This was done in the MOND (modified Newtonian dynamics) theory developed by Mordehai Milgrom in 1983 [15]. Rather than the familiar relationship

$$\begin{aligned} \nabla^2 \Phi_N &= 4\pi G \rho, \\ -\nabla \Phi_N &= \vec{a}, \end{aligned} \tag{6.2}$$

MOND suggests the form

$$\mu\left(\frac{|\vec{a}|}{a_0}\right)\vec{a} = -\nabla \Phi_N , \tag{6.3}$$

where a_0 is an acceleration scale and $\mu(x)$ is an undetermined function [15] which satisfies the conditions

$$\begin{aligned} \mu(x) &\approx x, \quad x << 1 \\ \mu(x) &\approx 1, \quad x >> 1 \end{aligned} . \tag{6.4}$$

In the everyday world $\frac{|\vec{a}|}{a_0} >> 1$, so $\mu\left(\frac{|\vec{a}|}{a_0}\right) \approx 1$ and equation (6.2) is upheld. For stars outside the galactic center $\frac{|\vec{a}|}{a_0} << 1$ and equation (6.3) becomes

$$\begin{aligned}
&\mu\left(\frac{|\vec{a}|}{a_0}\right)\vec{a} = -\nabla\Phi_N, \\
&\frac{|\vec{a}|}{a_0}\vec{a} = \frac{GM}{r^2}\hat{r}, \\
&\rightarrow |\vec{a}|^2 = \frac{GMa_0}{r^2} \\
&\rightarrow \frac{v^2}{r} = \frac{\sqrt{GMa_0}}{r} \\
&\rightarrow v = \sqrt[4]{GMa_0}.
\end{aligned}$$

Under this construction the velocity is found to be independent of the radius which is what we want. Although MOND is a great cosmological model, it is a poor theory of physics. One of its major flaws is that it is not consistent with momentum conservation. To prove this, consider a group of particles located in an inertial coordinate system. The modified Newton's second law takes the form

$$\sum_i F_i^{external} + \sum_{\substack{i,j \\ i\neq j}} F_i = \sum_i \mu\left(\frac{|\vec{a}_i|}{a_0}\right)\frac{d\vec{p}_i}{dt}.$$

If we assume that the only force acting on the particles is the gravitational force between the particles then Newton's third law is upheld and there are no external forces. This leaves

$$0 = \sum_i \mu\left(\frac{|\vec{a}_i|}{a_0}\right)\frac{d\vec{p}_i}{dt}.$$

For simplicity consider only two particles then

$$0 = \mu\left(\frac{|\vec{a}_1|}{a_0}\right)\frac{d\vec{p}_1}{dt} + \mu\left(\frac{|\vec{a}_2|}{a_0}\right)\frac{d\vec{p}_2}{dt}.$$

If the accelerations of both particles is large enough then this becomes:

$$\begin{aligned} 0 &= \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} \\ Const &= \vec{p}_1 + \vec{p}_2, \end{aligned}$$

and momentum is conserved under these conditions. In general, the acceleration of one particle may be larger than the MOND limit, while that of the other particle is smaller. In that case we get

$$0 = \frac{d\vec{p}_1}{dt} + \frac{|\vec{a}_2|}{a_0}\frac{d\vec{p}_2}{dt}.$$

$\vec{a}_2$ might vary with time so the time derivative cannot simply be pulled out from the right hand side to prove that the momentum is conserved. The formulation of MOND presented in equation (6.3) fails to preserve many of the conservation laws [16] including conservation of momentum. In order to pursue MOND as a viable alternative to dark matter it must uphold the conservation laws. It turns out that a classical action can be created which reproduces the MOND result and is consistent with the conservation laws [16]. This formulation is called AQUAL (A Quadratic Lagrangian) and it is based on the lagrangian

$$L = -\frac{a_0^2}{8\pi G} f\left(\frac{|\nabla\varphi|^2}{a_0^2}\right) - \rho\varphi \, , \qquad (6.5)$$

where φ is the modified gravitational potential, ρ is the mass density, and f is an undetermined function. This lagrangian has the corresponding field equation

$$\nabla \bullet \left[\mu\left(\frac{|\nabla\varphi|}{a_0{}^2}\right)\nabla\varphi \right] = 4\pi G\rho,$$

which can be written in terms of the Newtonian field φ_N by substituting in equation (6.2),

$$\nabla \bullet \left[\mu \left(\frac{|\nabla \varphi|}{a_0} \right) - \nabla \varphi_N \right] = 0. \tag{6.6}$$

AQUAL is a successful classical theory, but it is inconsistent with special relativity. A true modified theory gravity must be fully relativistic. One attempt at constructing a relativistic theory is by modifying general relativity such that the MOND result is achieved in the appropriate limit. The essence of general relativity is codified in the Einstein-Hilbert action

$$S = -\frac{1}{2k} \int R \sqrt{-g}\, d^4 x,$$

where $k = 8\pi G c^{-4}$, and R is the Ricci scalar curvature. This action reproduces the Einstein field equations and in the weak field limit the Einstein equations reduce to Newton's theory of gravitation. The idea is to modify the action so that the new field equations reproduce MOND results in the weak field limit. The easiest way to tamper with the Einstein-Hilbert action is to replace R with a function of R, and the action becomes

$$S = -\frac{1}{2k} \int f(R) \sqrt{-g}\, d^4 x. \tag{6.7}$$

Various functions may be chosen such that MOND results from a weak field approximation. Many theories have been proposed under this framework and some have failed because they allow for the existence of superluminal waves [17]. However, an even stronger argument against $f(R)$ theories was recently put forth by M. E. Soussa and R. P. Woodard [18]. Their work takes advantage of the experimental evidence discovered from the gravitational lensing of star light by galactic clusters. In particular it has been found that the degree to which star light is bent around an observed galactic cluster cannot be explained by the observed mass distribution [19]. This is seen as a victory for dark matter since the anomalous gravitational lensing can be explained away by adding extra matter to the cluster [19]. In order for modified gravity theories to survive, they must account for both the flat rotation curves of galaxies and the anomalous gravitational lensing. Woodard and Soussa [18] showed that any modified theory that is built on the metric alone cannot have a MOND like non-relativistic limit and account for the anomalous gravitational lensing. Since, the action shown in equation (6.7) is dependent only on the metric (see appendix 6) it cannot be used to generate acceptable modified theories of gravity.

To deal with these problems in modified theories of gravity J. D. Bekenstein and M. Milgrom [17] have proposed a theory which is consistent with both the rotation curve and lensing results. To address the problems introduced by Woodard and Soussa, Bekenstein (this summary of TeVeS follows [17]) uses an action which is dependent on a scalar field and a vector field as well as the metric tensor. The theory is appropriately named TeVeS (Tensor Vector Scalar Theory), and it has action principles for determining each parameter

$$S = S_g + S_s + S_v + S_m = \int \left(L_g + L_s + L_v + L_m \right) d^4x \,. \tag{6.8}$$

Here L_g is origin of the metric tensors dynamics, L_s is the origin of the scalar field's dynamics, L_v is the origin of the vector field's dynamics, and L_m describes any matter fields appearing in the theory. Following AQUAL's success, the scalar field is introduced so that TeVeS has a MONDian limit that preserves the conservation laws. The vector field is introduced to account for the anomalous gravitational lensing. The metric tensor lagrangian and the matter field lagrangian are kept so that general relativity is produced in the appropriate limit. L_g is given by the Einstein-Hilbert action

$$L_g = -\frac{1}{16\pi G} R\sqrt{-g} \ , \tag{6.9}$$

where we are using units such that $c = 1$. L_v is given by

$$L_v = -\frac{K}{32\pi G}\left[g^{\alpha\beta} g^{\mu\upsilon} \left(B_{\alpha\mu} B_{\beta\upsilon} \right) - 2\frac{\lambda}{K}\left(g^{\mu\upsilon} u_\mu u_\upsilon + 1 \right) \right]\sqrt{-g} \ , \tag{6.10}$$

where K is a dimensionless parameter and $B_{\alpha\beta} = \partial_\alpha u_\beta - \partial_\beta u_\alpha$, u^α is the time-directed unit vector field obeying $g^{\alpha\beta} u_{\alpha\beta} = -1$ and λ is a Lagrange multiplier constraining the vectors length to unity. L_s is given by

$$L_s = -\frac{1}{2}\left[\sigma^2 h^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi + \frac{1}{2}Gl^{-2}\sigma^4 F\left(kG\sigma^2\right)\right]\sqrt{-g}, \qquad (6.11)$$
$$h^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$$

where k is a dimensionless positive parameter, l is a length scale, F is an undetermined function, φ is a dynamic scalar field, and σ is a non-dynamical auxiliary scalar field. The matter field lagrangian is set to depend on field variables f and the physical metric $\bar{g}_{\alpha\beta}$,

$$L_m = L_m\left(\bar{g}_{\alpha\beta}, f^\alpha, f^\alpha{}_{;\mu}, \ldots\right)\sqrt{-\bar{g}}, \ . \qquad (6.12.a)$$

$$\bar{g}_{\alpha\beta} = e^{-2\varphi}\left(g_{\alpha\beta} + u_\alpha u_\beta\right) - e^{2\varphi}u_\alpha u_\beta. \qquad (6.12.b)$$

We saw in equation (6.6) that there is a relationship between the Newtonian field and the modified field. The physical metric behaves analogously as the modification to the Einstein metric. The physical metric represents the true spacetime metric and determines the geometrical spacetime structure. Variation of the total action with respect to the inverse metric gives the TeVeS Einstein equations for $g_{\alpha\beta}$,

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = 8\pi G\left(\bar{T}_{\alpha\beta} + \left(1 - e^{-4\varphi}\right)u^\mu \bar{T}_{\mu(\alpha}u_{\beta)} + \tau_{\alpha\beta}\right) + \theta_{\alpha\beta}, \qquad (6.13)$$
$$\bar{T}_{\mu(\alpha}u_{\beta)} = \bar{T}_{\mu\alpha}u_\beta + \bar{T}_{\mu\beta}u_\alpha,$$

where,

$$\tau_{\alpha\beta} = \sigma^2 \left[\partial_\alpha \varphi \partial_\beta \varphi - \frac{1}{2} g^{\mu\upsilon} \partial_\mu \varphi \partial_\upsilon \varphi g_{\alpha\beta} - u^\mu \partial_\mu \left(u_{(\alpha} \partial_\beta \varphi_{)} - \frac{1}{2} u^\upsilon \partial_\upsilon \varphi g_{\alpha\beta} \right) \right]$$
$$- \frac{1}{4} G l^{-2} \sigma^4 F\left(kG\sigma^2 \right) g_{\alpha\beta}, \qquad (6.14)$$

$$u_{(\alpha} \partial_\beta \varphi_{)} = u_\alpha \partial_\beta \varphi + u_\beta \partial_\alpha \varphi,$$

and

$$\theta_{\alpha\beta} = K \left(g^{\mu\upsilon} u_{[\mu,\alpha]} u_{[\upsilon,\beta]} - \frac{1}{4} g^{\sigma\tau} g^{\mu\upsilon} u_{[\sigma,\mu]} u_{[\tau,\upsilon]} g_{\alpha\beta} \right) - \lambda u_\alpha u_\beta,$$
$$u_{[\mu,\alpha]} = \partial_\mu u^\alpha - \partial_\alpha u^\mu. \qquad (6.15)$$

In the above equation, $\bar{T}_{\alpha\beta}$ is the usual stress energy tensor, $\tau_{\alpha\beta}$ is the analogous energy-momentum tensor resulting from the scalar field, and $\theta_{\alpha\beta}$ is the analogous energy-momentum tensor resulting from the vector field. Variation with respect to the inverse metric yields one field equation, however, there are four undetermined fields. TeVeS prescribes variation with respect to $\sigma, \varphi, u_\alpha$ as well. Variation with respect to the vector field yields,

$$K u^{[\alpha;\beta]}{}_{;\beta} + \lambda u^\alpha + 8\pi G \sigma^2 u^\beta \partial_\beta \varphi g^{\alpha\gamma} \partial_\gamma \varphi = 8\pi G \left(1 - e^{-4\varphi} \right) g^{\alpha\mu} u^\beta \bar{T}_{\mu\beta},$$
$$u^{[\alpha;\beta]} = u^\beta{}_{;\alpha} - u^\alpha{}_{;\beta}, \qquad (6.16)$$

where the semicolon represents a covariant derivative. Variation with respect to the dynamic scalar field gives,

$$\left[\mu\left(kl^2h^{\mu\upsilon}\partial_\mu\varphi\partial_\upsilon\varphi\right)h^{\alpha\beta}\partial_\alpha\varphi\right]_{;\beta}=kG\left[g^{\alpha\beta}+\left(1+e^{-4\varphi}\right)u^\alpha u^\beta\right]\bar{T}_{\alpha\beta}, \tag{6.17}$$

where μ is a function that is related to the undetermined function F via the relationship,

$$-\mu(y)F(\mu)-\frac{1}{2}\mu^2F'(\mu)=y.$$

Variation of the non-dynamical scalar field yields the equation,

$$-kG\sigma^2F-\frac{1}{2}\left(kG\sigma^2\right)^2F'=kl^2h^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi. \tag{6.18}$$

TeVeS spacetime has a definite geometrical structure which is determined by the physical metric. It also uses the stress energy tensor to codify all information regarding the matter-energy distribution in the spacetime. It follows the same general concept of general relativity that (a measure of local spacetime curvature) = (a measure of matter energy density) [20] and so it should be subject to the same thermodynamic argument presented in section (5). If we could isolate an expression for the stress energy tensor in terms of the metric tensor, vector, and scalar fields then we could substitute that expression into the left hand side of equation (5.8) and check to see if it reduces to the right hand side. Equation (6.13) looks promising since it already incorporates the

Einstein tensor. However, we must remember that the TeVeS spacetime structure is determined by the physical metric. This means that the Ricci tensor is now built out of the physical metric and is in general different then the Ricci tensor which is built on the Einstein metric. The Ricci tensor seen in equation (5.8) then becomes the physical Ricci tensor in the TeVeS setting. The tensor contained in the Einstein tensor of equation (6.13) is the non-physical Ricci tensor (the Einstein Ricci tensor) [17]. This makes the technique of substituting (6.13) into (5.8) more complicated. We could proceed by writing the physical Ricci tensor in terms of the Einstein metric by using equation (6.12.b) and hope that there is a simple relationship between the physical Ricci tensor and the Einstein Ricci tensor. There are also terms in (6.13) that are independent of the metric and we would have to determine what happens to them when they are contracted with the null $K^{\alpha}K^{\beta}$ vectors. Most importantly, the dynamics of the four TeVeS fields are determined by the complicated partial differential equations (6.13), (6.16), (6.17), and (6.18). The stress energy tensor cannot be extracted from these equations as desired. However, we can specify a particular stress energy tensor and determine the corresponding TeVeS fields. Then we can check TeVeS's thermodynamic validity under that restricted case.

7. Conclusion

General Relativity passes all solar system tests with flying colors, but fails to describe the motion of observable matter on the galactic scale. The conservative approach in addressing this issue is to preserve the gravitational theory while proposing the existence of additional unobservable matter. The more radical approach is to propose an alternative theory of gravity. The work of Woodard and Soussa [18] indicates that any theory that aims to explain both the flat galactic rotation curves and the anomalous gravitational lensing cannot be built on the metric alone. It is possible for gravitational theories to be built out of non-metric components along with metric components as was done in TeVeS theory. The thermodynamic argument developed in chapter (5) should be applicable so long as gravitational theories operate in a spacetime that is represented by a Riemannian manifold. The thermodynamic argument can then be used to check new theories of gravity for their consistency with thermodynamics. This may serve as a guiding principle for weeding out unacceptable theories. In the case of TeVeS more work must be done in testing its thermodynamic validity for specific choices of the stress energy tensor. This is still a daunting task considering the complexity of the TeVeS field equations. A starting point is to choose a zero mass-energy distribution so that the stress energy tensor disappears from the field equations. This also reduces the Einstein metric to the Minkowski metric which makes computations significantly simpler since the off diagonal elements are zero. Using the field equations we can determine the vector and scalar fields and substitute them into (6.12.b). The physical Ricci tensor can then be written in terms of the Einstein metric as well as the vector and scalar fields. Then we can check if the contraction of the physical Ricci tensor with the null vectors in equation (5.8) yields zero.

Appendix

Appendix 1

Appendix.1 follows [5, pg-558]. We can compute the classical Euler-Lagrange equations for the Lagrangian given by equation (2.1).

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_j}\right)-\frac{\partial L}{\partial \eta_j}=0$$

$$\frac{d}{dt}\left(\frac{\partial\left(\sum_i \frac{1}{2} l\left[\frac{m}{l}\left(\dot{\eta}_i\right)^2-kl\left(\frac{\eta_{i+1}-\eta_i}{l}\right)^2\right]\right)}{\partial \dot{\eta}_j}\right)-\frac{\partial\left(\sum_i \frac{1}{2} l\left[\frac{m}{l}\left(\dot{\eta}_i\right)^2-kl\left(\frac{\eta_{i+1}-\eta_i}{l}\right)^2\right]\right)}{\partial \eta_j}=0.$$

We can divide through by l , and realizing that the sums only survive for $i=j$ the equation of motion becomes

$$\begin{aligned} &\frac{m}{l}\frac{d^2\eta_j}{dt^2}-kl\left(\frac{\eta_{j+1}-\eta_j}{l^2}\right)+kl\left(\frac{\eta_j-\eta_{j-1}}{l^2}\right)=0 \\ &m\frac{d^2\eta_j}{dt^2}-kl\left[\left(\frac{\eta_{j+1}-\eta_j}{l}\right)-\left(\frac{\eta_j-\eta_{j-1}}{l}\right)\right]=0. \end{aligned} \qquad \text{(a.1.1)}$$

This is equivalent to Newton's second law, where the second term is the force term. For an elastic rod obeying Hooke's law, the extension per unit length is directly proportional to the force acting in the rod.

$$F = Y\zeta, \tag{a.1.2}$$

where Y is Young's modulus and ζ is the extension per unit length. The extension per unit length in the rod is given in the bracketed term of equation (a.1.1). By comparing (a.1.1) with (a.1.2) we see that for the continuous rod kl corresponds to Young's modulus Y.

Appendix 2

To check the Lorentz invariance of (2.5), we recall the boost transformations for inertial reference frames in Minkowski space

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right), \\ x' &= \gamma\left(x - vt\right). \end{aligned} \tag{a.2.1}$$

A scalar is specified by one real number and is a quantity that doesn't change under coordinate transformations. In relativity theory, inertial reference frames are coordinate frames in Minkowski space. Thus, between any two inertial reference frames in Minkowski space (say a primed and unprimed frame), the scalar field obeys the condition

$$\varphi(x,t)=\varphi'(x',t').$$

In order to show that equation (2.5) is Lorentz invariant we start by transforming $\frac{\partial\varphi}{\partial t},\frac{\partial\varphi}{\partial x}$,

$$\frac{\partial\varphi(x,t)}{\partial t}=\frac{\partial\varphi(x',t')}{\partial t}.$$

By the chain rule we have

$$\frac{\partial\varphi(x,t)}{\partial t}=\frac{\partial\varphi(x',t')}{\partial t}=\frac{\partial\varphi}{\partial x'}\frac{\partial x'}{\partial t}+\frac{\partial\varphi}{\partial t'}\frac{\partial t'}{\partial t}.$$

Substituting in the Lorentz transformations this becomes

$$\begin{aligned}\frac{\partial\varphi}{\partial x'}\frac{\partial x'}{\partial t}+\frac{\partial\varphi}{\partial t'}\frac{\partial t'}{\partial t}&=\frac{\partial\varphi}{\partial x'}\frac{\partial}{\partial t}\left(\gamma\left(x-vt\right)\right)+\frac{\partial\varphi}{\partial t'}\frac{\partial}{\partial t}\left(\gamma\left(t-\frac{vx}{c^2}\right)\right)\\&=\left(-v\gamma\right)\frac{\partial\varphi}{\partial x'}+\gamma\frac{\partial\varphi}{\partial t'}\ .\end{aligned}$$

Then,

$$\begin{aligned}\left(\frac{\partial\varphi(x',t')}{\partial t}\right)^2&=\left(\left(-v\gamma\right)\frac{\partial\varphi}{\partial x'}+\gamma\frac{\partial\varphi}{\partial t'}\right)^2\\&=v^2\gamma^2\left(\frac{\partial\varphi}{\partial x'}\right)^2-2v\gamma^2\frac{\partial\varphi}{\partial x'}\frac{\partial\varphi}{\partial t'}+\gamma^2\left(\frac{\partial\varphi}{\partial t'}\right)^2.\end{aligned}\qquad\text{(a.2.2)}$$

Next,

$$\frac{\partial\varphi(x,t)}{\partial x}=\frac{\partial\varphi(x',t')}{\partial x}=\frac{\partial\varphi}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial\varphi}{\partial t'}\frac{\partial t'}{\partial x}$$
$$=\frac{\partial\varphi}{\partial x'}\frac{\partial}{\partial x}\left(\gamma\left(x-vt\right)\right)+\frac{\partial\varphi}{\partial t'}\frac{\partial}{\partial x}\left(\gamma\left(t-\frac{vx}{c^2}\right)\right)$$
$$=\gamma\frac{\partial\varphi}{\partial x'}+\left(-\frac{\gamma v}{c^2}\right)\frac{\partial\varphi}{\partial t'}$$

(a.2.3)

$$\rightarrow -c^2\left(\frac{\partial\varphi(x,t)}{\partial x}\right)^2=-c^2\left(\gamma\frac{\partial\varphi}{\partial x'}+\left(-\frac{\gamma v}{c^2}\right)\frac{\partial\varphi}{\partial t'}\right)^2$$
$$=-c^2\left(\gamma^2\left(\frac{\partial\varphi}{\partial x'}\right)^2-2\frac{\gamma^2 v}{c^2}\frac{\partial\varphi}{\partial x'}\frac{\partial\varphi}{\partial t'}+\frac{\gamma^2 v^2}{c^4}\left(\frac{\partial\varphi}{\partial t'}\right)^2\right)$$
$$=-c^2\gamma^2\left(\frac{\partial\varphi}{\partial x'}\right)^2+2\gamma^2 v\frac{\partial\varphi}{\partial x'}\frac{\partial\varphi}{\partial t'}-\frac{\gamma^2 v^2}{c^2}\left(\frac{\partial\varphi}{\partial t'}\right)^2.$$

Combining (a.2.2) and (a.2.3) we have,

$$\left(\frac{\partial\varphi(x,t)}{\partial t}\right)^2-c^2\left(\frac{\partial\varphi(x,t)}{\partial x}\right)^2=\left(\frac{\partial\varphi(x',t')}{\partial t}\right)^2-c^2\left(\frac{\partial\varphi(x',t')}{\partial x}\right)^2$$
$$=v^2\gamma^2\left(\frac{\partial\varphi}{\partial x'}\right)^2-2v\gamma^2\frac{\partial\varphi}{\partial x'}\frac{\partial\varphi}{\partial t'}+\gamma^2\left(\frac{\partial\varphi}{\partial t'}\right)^2-c^2\gamma^2\left(\frac{\partial\varphi}{\partial x'}\right)^2+2\gamma^2 v\frac{\partial\varphi}{\partial x'}\frac{\partial\varphi}{\partial t'}-\frac{\gamma^2 v^2}{c^2}\left(\frac{\partial\varphi}{\partial t'}\right)^2$$
$$=\gamma^2\left(v^2-c^2\right)\left(\frac{\partial\varphi}{\partial x'}\right)^2+\gamma^2\left(1-\frac{v^2}{c^2}\right)\left(\frac{\partial\varphi}{\partial t'}\right)^2$$
$$=\gamma^2\left(1-\frac{v^2}{c^2}\right)\left(\frac{\partial\varphi}{\partial t'}\right)^2-c^2\gamma^2\left(1-\frac{v^2}{c^2}\right)\left(\frac{\partial\varphi}{\partial x'}\right)^2$$
$$=\left(\frac{\partial\varphi}{\partial t'}\right)^2-c^2\left(\frac{\partial\varphi}{\partial x'}\right)^2.$$

(a.2.4)

The integrand is Lorentz invariant. Now we must check the differential form $dxdt$.

Rearranging equation (a.2.1) we find that

$$\begin{aligned} t &= \gamma\left(t' + \frac{vx'}{c^2}\right) \\ x &= \gamma\left(x' + vt'\right), \\ \rightarrow dt &= \gamma\left(dt' + \frac{vdx'}{c^2}\right) \\ dx &= \gamma\left(dx' + vdt'\right). \end{aligned} \tag{a.2.5}$$

The differential form $dxdt$ found in the action integral is the result of the exterior product $dx \wedge dt$. The exterior product is an anti-symmetric product which obeys the following rules:

$$\begin{aligned} dx \wedge dy &= -dy \wedge dx \\ dx \wedge dx &= 0. \end{aligned}$$

From the transformations found in equation (a.2.5) we see that

$$
\begin{aligned}
dxdt = dx \wedge dt = \gamma\left(dx' + vdt'\right) \wedge \gamma\left(dt' + \frac{v}{c^2}dx'\right) \\
&= \gamma^2 dx' \wedge dt' + \gamma^2 \frac{v^2}{c^2} dx' \wedge dx' + \gamma^2 vdt' \wedge dt' + \gamma^2 \frac{v^2}{c^2} dt' \wedge dx' \\
&= \gamma^2 dx' \wedge dt' + \gamma^2 \frac{v^2}{c^2} dt' \wedge dx' \\
&= \gamma^2 dx' \wedge dt' - \gamma^2 \frac{v^2}{c^2} dx' \wedge dt' \\
&= \gamma^2 \left(1 - \frac{v^2}{c^2}\right) dx' \wedge dt' \\
&= dx' \wedge dt',
\end{aligned}
$$

or hiding the exterior product

$$dxdt = dx'dt'. \qquad \text{(a.2.6)}$$

Combining (a.2.4), and (a.2.6) we see that the action arrived at in equation (2.5) is indeed Lorentz invariant

$$S(\varphi) = \int_0^T \int \frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial t}\right)^2 - c^2\left(\frac{\partial\varphi}{\partial x}\right)^2\right] dxdt = \int_0^{T'} \int \frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial t'}\right)^2 - c^2\left(\frac{\partial\varphi}{\partial x'}\right)^2\right] dx'dt'. \qquad \text{(a.2.7)}$$

Appendix 3

This is a non rigorous sketch of the coordinate invariance of the action

$$S = \int dx^4 \sqrt{-g}\,\frac{1}{2}\left(g^{\mu\upsilon}\partial_\mu\varphi\partial_\upsilon\varphi - m^2\varphi^2\right).$$

The scalar field φ is by definition coordinate invariant so we need to show that the first term is also invariant under an arbitrary coordinate transformation.

$$x^i = \left(x^0, x^1, x^2, x^3\right) \rightarrow x'^i = \left(x'^0, x'^1, x'^2, x'^3\right).$$

The inverse metric tensor transforms according to the rule

$$g^{\mu\upsilon} = \frac{\partial x^\mu}{\partial x'^\lambda}\frac{\partial x^\upsilon}{\partial x'^\sigma} g'^{\lambda\sigma}. \qquad \text{(a.3.1)}$$

The partial derivatives of the field transform according to the chain rule

$$\partial_\mu \varphi(x) = \frac{\partial x'^\lambda}{\partial x^\mu}\frac{\partial \varphi(x')}{\partial x'^\lambda} = \frac{\partial x'^\lambda}{\partial x^\mu}\partial'_\lambda \varphi'(x'). \qquad \text{(a.3.2)}$$

Then,

$$\begin{aligned} g^{\mu\upsilon}\partial_\mu \varphi(x)\partial_\upsilon \varphi(x) &= \frac{\partial x^\mu}{\partial x'^\lambda}\frac{\partial x^\upsilon}{\partial x'^\sigma} g'^{\lambda\sigma}\frac{\partial x'^\lambda}{\partial x^\mu}\partial'_\lambda \varphi'(x')\frac{\partial x'^\sigma}{\partial x^\upsilon}\partial'_\sigma \varphi'(x') \\ &= \frac{\partial x^\mu}{\partial x'^\lambda}\frac{\partial x^\upsilon}{\partial x'^\sigma}\frac{\partial x'^\lambda}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\upsilon} g'^{\lambda\sigma}\partial'_\lambda \varphi'(x')\partial'_\sigma \varphi'(x') \qquad \text{(a.3.3)} \\ &= g'^{\lambda\sigma}\partial'_\lambda \varphi'(x')\partial'_\sigma \varphi'(x'), \end{aligned}$$

is also coordinate invariant (this is of course not mathematically rigorous). It remains to show that $dx^4\sqrt{-g}$ is also invariant. A direct approach to this is significantly more involved than the other steps, so I will only provide a sketch of how to proceed by brute force. The metric with components $g_{\mu\upsilon}$ can be thought of as a matrix with determinant

$$g = \det g_{\mu\upsilon} = \begin{vmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{vmatrix} = \sum_{i,j,k,l} \varepsilon_{ijkl} g_{0i} g_{1j} g_{2k} g_{3l}, \qquad \text{(a.3.4)}$$

where ε_{ijkl} is a four component Levi-Civita symbol. The symbol takes the value of 0 for repeated indices, 1 for even permutations, and -1 for odd permutations. Terms with repeated indices are not considered in the sum. This leaves 24 possible terms

$$\begin{aligned}
&\varepsilon_{0123} = 1,\ \varepsilon_{0132} = -1,\ \varepsilon_{0213} = -1,\ \varepsilon_{0231} = 1,\ \varepsilon_{0312} = 1,\ \varepsilon_{0321} = -1 \\
&\varepsilon_{1023} = -1,\ \varepsilon_{1032} = 1,\ \varepsilon_{1203} = 1,\ \varepsilon_{1230} = -1,\ \varepsilon_{1302} = -1,\ \varepsilon_{1320} = 1 \\
&\varepsilon_{2103} = -1,\ \varepsilon_{2130} = 1,\ \varepsilon_{2013} = 1,\ \varepsilon_{2031} = -1,\ \varepsilon_{2310} = -1,\ \varepsilon_{2301} = 1 \\
&\varepsilon_{3120} = 1,\ \varepsilon_{3102} = -1,\ \varepsilon_{3210} = -1,\ \varepsilon_{3201} = 1,\ \varepsilon_{3012} = 1,\ \varepsilon_{3021} = -1\,.
\end{aligned}$$

So the sum in (a.3.4) looks like

$$\sum_{i,j,k,l} \varepsilon_{ijkl} g_{0i} g_{1j} g_{2k} g_{3l} = g_{00} g_{11} g_{22} g_{33} - g_{00} g_{11} g_{23} g_{32} - g_{00} g_{12} g_{21} g_{33} + \ldots . \qquad \text{(a.3.5)}$$

Some simplification can be made by imposing the condition that the metric is symmetric. Each metric component transforms according to the sums

$$g_{\mu\upsilon} = g'_{\lambda\sigma} \frac{\partial x'^{\lambda}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\upsilon}}. \quad \text{(a.3.6)}$$

This must be substituted for each component in (a.3.5). Next we notice that the integration element transforms as

$$dx^{\mu} = \frac{\partial x^{\mu}}{\partial x'^{\upsilon}} dx'^{\upsilon}. \quad \text{(a.3.7)}$$

Substituting (a.3.6) and (a.3.7) into $dx^4\sqrt{-g}$ yields a long and troublesome expression. However, it yields sufficient cancelation to reduce to

$$dx^4\sqrt{-g} = dx'^4\sqrt{-g'}. \quad \text{(a.3.8)}$$

The odd term $\sqrt{-g}$ is introduced as a scaling factor to ensure invariance in the hyper-volume element. With regards to four dimensional spacetime metrics with the conventional (-, +, +, +) signature the determinant is always negative, so we include the negative sign under the square root to ensure that the action is real.

Appendix 4

The four velocity of the accelerated observer is given by

$$u = \frac{d}{d\tau}[ct, x] = \left[c\gamma, \frac{dx}{d\tau}\right] = \left[c\gamma, \frac{dx}{dt}\frac{dt}{d\tau}\right] = [c\gamma, v\gamma].$$

The four acceleration of the observer is then given by

$$\begin{aligned}
a = \frac{du}{d\tau} &= \frac{d}{d\tau}[c\gamma, v\gamma] \\
&= \left[c\frac{d\gamma}{d\tau}, v\frac{d\gamma}{d\tau} + \gamma\frac{dv}{d\tau}\right] \\
&= \left[c\frac{d}{d\tau}\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}, v\frac{d}{d\tau}\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} + \gamma\frac{dv}{dt}\frac{dt}{d\tau}\right] \\
&= \left[c\left(\frac{-1}{2}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left(\frac{-2v}{c^2}\right)\frac{dv}{d\tau}\right), v\left(\frac{-1}{2}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left(\frac{-2v}{c^2}\right)\frac{dv}{d\tau}\right) + \gamma^2\ddot{x}\right]
\end{aligned}$$

$$a=\left[\frac{1}{c}\gamma^3 v\frac{dv}{dt}\frac{dt}{d\tau},\gamma^3 v^2\frac{1}{c^2}\frac{dv}{dt}\frac{dt}{d\tau}+\gamma^2\ddot{x}\right]=\left[\frac{1}{c}\gamma^4 v\ddot{x},\gamma^4 v^2\frac{1}{c^2}\ddot{x}+\gamma^2\ddot{x}\right]$$

$$=\left[\frac{1}{c}\gamma^4 v\ddot{x},\left(\gamma^4 v^2\frac{1}{c^2}+\gamma^2\right)\ddot{x}\right]=\left[\frac{1}{c}\gamma^4 v\ddot{x},\left(\frac{v^2}{c^2\left(1-\frac{v^2}{c^2}\right)^2}+\frac{1}{\left(1-\frac{v^2}{c^2}\right)}\right)\ddot{x}\right]$$

$$=\left[\frac{1}{c}\gamma^4 v\ddot{x},\left(\frac{v^2+c^2\left(1-\frac{v^2}{c^2}\right)}{c^2\left(1-\frac{v^2}{c^2}\right)^2}\right)\ddot{x}\right]=\left[\frac{1}{c}\gamma^4 v\ddot{x},\left(\frac{v^2-v^2+c^2}{c^2\left(1-\frac{v^2}{c^2}\right)^2}\right)\ddot{x}\right]$$

$$=\left[\frac{1}{c}\gamma^4 v\ddot{x},\gamma^4\ddot{x}\right]$$
$$=\gamma^4\ddot{x}\left[\frac{1}{c}v,1\right].$$

Then the magnitude of the acceleration is

$$|a|=\sqrt{a\bullet a}=\gamma^4\ddot{x}\sqrt{\left[\frac{v}{c},1\right]\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{bmatrix}\frac{v}{c}\\ 1\end{bmatrix}}$$
$$=\gamma^4\ddot{x}\frac{1}{\gamma}$$
$$=\gamma^3\ddot{x}.$$

So we have,

$$|a| = \frac{dv}{dt}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}} \rightarrow \int |a|\,dt = \int \left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}} dv$$

$$|a|t + k = \int \left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}} dv, \qquad let\, u^2 = \frac{v^2}{c^2}, c du = dv, then$$

$$|a|t + k = c\int \left(1-u^2\right)^{-\frac{3}{2}} du, \qquad let\, u^2 = \sin^2\theta, du = \cos\theta d\theta,\ then$$

$$|a|t + k = c\int \left(1-\sin^2\theta\right)^{-\frac{3}{2}} \cos\theta d\theta = c\int \frac{1}{\cos^2\theta} d\theta = c\frac{\sin\theta}{\cos\theta}$$

$$= \frac{v}{\sqrt{1-\frac{v^2}{c^2}}}$$

If $v = 0$ *at* $t = 0$, *then* $k = 0$

$$|a|^2 t^2 = \frac{v^2}{1-\frac{v^2}{c^2}}$$

$$v(t) = \frac{|a|t}{\sqrt{1+\frac{|a|^2 t^2}{c^2}}}.$$

Next,

$$dt = \frac{d\tau}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Substituting in our expression for $v(t)$ we have,

$$\int\left(1-\frac{|a|^2 t^2}{c^2+|a|^2 t^2}\right)^{\frac{1}{2}} dt = \int d\tau$$

$$\int\left(\frac{c^2}{c^2+|a|^2 t^2}\right)^{\frac{1}{2}} dt = \tau + k$$

$$let\ \ u = \frac{|a|}{c}t,\ du = \frac{|a|}{c}dt$$

$$\frac{c}{|a|}\int\frac{1}{\sqrt{1+u^2}}du = \tau + k$$

$$\frac{c}{|a|}\sinh^{-1}(u) = \tau + k$$

$$If\ \ \tau = 0,\ when\ \ t = 0,\ then\ \ k = 0$$

$$\rightarrow t = \frac{c}{|a|}\sinh\left(\frac{|a|}{c}\tau\right).$$

Substituting this into our expression for $v(t)$ we get

$$v(t) = c\tanh\left(\frac{|a|\tau}{c}\right)$$

$$\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{dx}{d\tau}\frac{1}{\gamma} = c\tanh\left(\frac{|a|\tau}{c}\right)$$

$$\rightarrow dx = \gamma c\tanh\left(\frac{|a|\tau}{c}\right)d\tau$$

$$dx = c\frac{1}{\sqrt{1-\tanh^2\left(\frac{|a|\tau}{c}\right)}}\tanh\left(\frac{|a|\tau}{c}\right)d\tau = \frac{c\tanh\left(\frac{|a|\tau}{c}\right)}{\sec h\left(\frac{|a|\tau}{c}\right)}d\tau\ = c\sinh\left(\frac{|a|\tau}{c}\right)d\tau$$

Integration Gives

$$\rightarrow \qquad x(\tau) = \frac{c^2}{|a|}\cosh\left(\frac{|a|\tau}{c}\right).$$

Appendix 5

In order to derive equation (3.31) we start with the commutation relation

$$\left[b(\Omega), b^{\dagger}(\Omega')\right] = \delta(\Omega - \Omega'). \qquad \text{(a.5.1)}$$

From the Bogoliubov transformations (equation (3.23)), equation (a.5.1) becomes

$$\begin{aligned}
\delta(\Omega-\Omega') &= \left[\int_0^\infty d\omega\left(\alpha_{\omega\Omega}a(\omega) - \beta_{\omega\Omega}a^{\dagger}(\omega)\right), \int_0^\infty d\omega'\left(\alpha^{*}{}_{\omega'\Omega'}a^{\dagger}(\omega') - \beta^{*}{}_{\omega'\Omega'}a(\omega')\right)\right] \\
&= \int_0^\infty\int_0^\infty d\omega d\omega'\left(\alpha_{\omega\Omega}\alpha^{*}{}_{\omega'\Omega'}\delta(\omega-\omega') - \beta_{\omega\Omega}\beta^{*}{}_{\omega'\Omega'}\delta(\omega-\omega')\right) \\
&= \int_0^\infty d\omega\left(\alpha_{\omega\Omega}\alpha^{*}{}_{\omega\Omega'} - \beta_{\omega\Omega}\beta^{*}{}_{\omega\Omega'}\right).
\end{aligned}$$

Appendix 6

Consider the modified action

$$S = -\frac{1}{2k}\int f(R)\sqrt{-g}\,d^4x.$$

g is the determinant of the metric so it is built out of metric components. The Ricci scalar curvature R is determined by the contraction of the inverse metric tensor with the Ricci tensor

$$R = g^{\mu\upsilon} R_{\mu\upsilon} .$$

The Ricci tensor is a sort of sub tensor of the Riemann curvature tensor such that

$$R_{\mu\upsilon} = R^{\lambda}{}_{\mu\lambda\upsilon} .$$

The Riemann tensor built out of the sum of derivatives of Christoffel symbols

$$R^{\lambda}{}_{\mu\upsilon\sigma} = \partial_{\upsilon}\Gamma^{\lambda}{}_{\mu\sigma} - \partial_{\sigma}\Gamma^{\lambda}{}_{\mu\upsilon} + \Gamma^{\eta}{}_{\mu\sigma}\Gamma^{\lambda}{}_{\eta\upsilon} - \Gamma^{\eta}{}_{\mu\upsilon}\Gamma^{\lambda}{}_{\eta\sigma} .$$

Lastly, the Christoffel symbols are built out of derivatives of the metric tensor

$$\Gamma^{\lambda}{}_{\mu\upsilon} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_{\mu} g_{\sigma\upsilon} + \partial_{\upsilon} g_{\sigma\mu} - \partial_{\sigma} g_{\mu\upsilon} \right).$$

Ultimately R is dependent on the components of the metric tensor and so the original action is dependent on the metric only.

Appendix 7

In special relativity there is no preferred inertial frame and the field can be determined from the action given in equation (2.6). It is natural to choose the proper time of each inertial observer as the time coordinate for the field expansion. This is because the time measured by a clock that is commoving with an inertial observer is defined to be the proper time of that observer. For accelerated motion and motion through curved spacetime the proper time is also a natural choice of time coordinate for the same reason. In section (3) we choose the Rindler coordinates as effective coordinates for the accelerated observer because ξ^0 coincides with the observer's proper time when evaluated on the observer's world line. Effective coordinates must also be well defined in the region of spacetime that we are concerned with. Although, the Rindler coordinates only cover one quarter of the (1+1) dimensional spacetime, they successfully map the region of constant acceleration and so they are effective coordinates. The same reasons can be provided for the choice of coordinates used in section (4). The analysis of quantum field theory in the presence of a Schwarzschild black hole is simplified by the spherical symmetry of the space time. In an arbitrarily curved spacetime, there may not be an obvious choice of time coordinate. How to proceed in cases where there is no spacetime symmetry is a question that is yet to be answered.

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Vita

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